Math 565: Functional Analysis
Lecture 10

Existence of bdd linear functionals: Hahn-Banach theorems.

It's easy to build linear functionals on ∞-dim spaces because we can take any function on a linear (Hamel) basis and uniquely extend it to a linear functional, but this typically won't be continuous.

Def. For a real vector space X, a **sublinear functional** is a function $p: X \to \mathbb{R}$ satisfying:
(i) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$;
(ii) $p(\lambda x) = \lambda p(x)$ for all $\lambda \geq 0$ and $x \in X$. (In particular, $p(0_X) = 0$.)

Examples. All linear functionals and all semi-norms are sublinear functionals.

Real Hahn-Banach. Let $Y \subseteq X$ be real vector spaces, p a sublinear functional on X, and f a linear functional on Y satisfying $f \leq p|_Y$, i.e. $f(y) \leq p(y)$ for all $y \in Y$. Then f admits an extension to a linear functional $\tilde{f}: X \to \mathbb{R}$ satisfying $\tilde{f} \leq p$.

Remark. Let p be a semi-norm. Then $f(y) \leq p(y)$ and $-f(y) = f(-y) \leq p(-y) = p(y)$ implies

$$|f(y)| \leq p(y).$$

Thus, the theorem will yield a functional $\tilde{f}$ whose p norm is ≤ 1.

Proof (AC). We start with Y a extend "one dimension at a time" then apply Zorn's lemma. First, let's show that we can always extend by one dimension. Let $x \in X \setminus Y$ and we define an extension $g: Y + \mathbb{R}x \to \mathbb{R}$ of f satisfying $g \leq p|_{Y+\mathbb{R}x}$. This extension is uniquely determined by the value $g(x)$ by linearity, so it remain to chose this value so $g \leq p|_{Y+\mathbb{R}x}$. Note that if this was satisfied, we would have to have: for all $y_1, y_2 \in Y$,
(i) $f(y_1) - g(x) = g(y_1 - x) \leq p(y_1 - x)$, so $g(x) \geq f(y_1) - p(y_1 - x)$.

(ii) $f(y_2)+g(x)=g(y_2+x)\le p(y_2+x)$, so $g(x)\le p(y_2+x)-f(y_2)$.

Hence, (i) + (ii) gives: $\forall y_1,y_2\in Y$,

$$f(y_1)-p(y_1-x)\le g(x)\le p(y_2+x)-f(y_2).$$

Thus, we need to have

$$\sup_{y_1\in Y} f(y_1)-p(y_1-x)\le g(x)\le \inf_{y_2\in Y} p(y_2+x)-f(y_2). \quad (*)$$

But $f(y_1)-p(y_1-x)\le p(y_2+x)-f(y_2)$ because $f(y_1)+f(y_2)=f(y_1+y_2)\le p(y_1+y_2)=$ $=p(y_1-x+x+y_2)\le p(y_1-x)+p(x+y_2)$. Hence we may take a value of $g(x)$ to satisfy $(*)$.

It remains to check that $g\le p|_{Y+\mathbb{R}x}$. Fix $y\in Y$ and $\lambda>0$, and compute:

(i) $g(y-\lambda x)=f(y)-\lambda g(x)\le f(y)-\lambda\left(f(\tfrac{1}{\lambda}y)-p(\tfrac{1}{\lambda}y-x)\right)=f(y)-\lambda\cdot\tfrac{1}{\lambda}f(y)+p(y-\lambda x)=p(y-\lambda x)$.

(ii) $g(y+\lambda x)=f(y)+\lambda g(x)\le f(y)+\lambda\left(p(\tfrac{1}{\lambda}y+x)-f(\tfrac{1}{\lambda}y)\right)=f(y)-\tfrac{1}{\lambda}\lambda f(y)+p(y+\lambda x)=p(y+\lambda x)$.

Now we apply Zorn's lemma and get an extension with maximal domain, which then has to be X because otherwise we could extend further by one more dimension. Here are the details.

Zorn's lemma ($\Leftrightarrow$ Axiom of Choice). Let $(P,\le)$ be a partial order such that every linearly ordered subset $C\subseteq P$ has an upper bound. Then P has a maximal element $m\in P$, i.e. $\nexists p\in P$ with $m<p$.

In our case $P:=\{g:Z\to\mathbb{R}: Z\subseteq X \text{ subspace}, g \text{ linear}, g\le p|_Z, f\subseteq g\}$ and $g_1\le g_2 :\Leftrightarrow g_1\subseteq g_2$ (functions are just sets of pairs). Then if $C\subseteq P$ is a linearly ordered set of such functionals, then $u:=\bigcup C=\bigcup_{g\in C} g$ is a linear functional on $Z:=\bigcup_{g\in C}\operatorname{dom}(g)$ satisfying $u\le p|_Z$ (check!), so $u\in P$ and $g\subseteq u$ for all $g\in C$, hence u is an upper bound of C, in fact, the least upper bound.

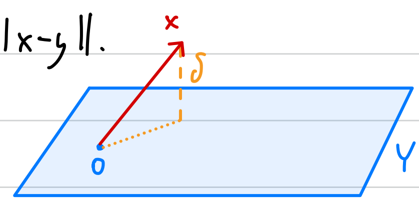
Thus, Zorn's lemma applies and gives a maximal $\tilde{f}\in P$. Then $\operatorname{dom}(\tilde{f})$ must be all of X since otherwise, taking $x\in X\setminus\operatorname{dom}(\tilde{f})$, we can extend $\tilde{f}$ further to $h:\operatorname{dom}(\tilde{f})+\mathbb{R}x\to\mathbb{R}$ satisfying $h\in P$, as demonstrated above, contradicting the maximality of $\tilde{f}$. $\boxed{\text{QED}}$

Complex Hahn-Banach. Let X be a complex vector space and p be a semi-norm on X. Let $Y \subseteq X$ be a subspace and f a linear functional on Y satisfying $|f(y)| \le p(y)$ for all $y \in Y$. Then f admits an extension to a linear functional $\hat{f}: X \to \mathbb{C}$ satisfying $|\hat{f}| \le p$.

Proof. Let $u := \operatorname{Re} f$, so u is a real linear functional on Y satisfying $|u(y)| \le |f(y)| \le p(y)$ for all $y \in Y$, hence $u(y) \le p(y)$ $\forall y \in Y$. By real Hahn-Banach, there is an extension $\tilde{u}: X \to \mathbb{R}$ of u satisfying $\tilde{u} \le p$. As in the remark above, $\tilde{u}(-x) \le p(-x) = p(x)$ and $\tilde{u}(x) \le p(x)$ implies $|\tilde{u}(x)| \le p(x)$, so $|\tilde{u}| \le p$. Take $\hat{f}(x) := \tilde{u}(x) - i\tilde{u}(ix)$ for all $x \in X$. This is linear, by an earlier lemma, and $\|\hat{f}\|_p = \|\tilde{u}\|_p$; more precisely, for each $x \in X$, take $\alpha := \overline{\operatorname{sgn} \hat{f}(x)}$, so $|\hat{f}(x)| = \alpha \cdot \hat{f}(x) = \hat{f}(\alpha x) = \tilde{u}(\alpha x) \le p(\alpha x) = |\alpha| \cdot p(x) = p(x)$. $\boxed{\text{QED}}$

Main Hahn-Banach corollaries. Let X be a (complex) normed vector space.

(a) For each closed subspace $Y \subseteq X$ and $x \in X \setminus Y$, $\exists f \in X^*$ with $f|_Y = 0$, $f(x) \neq 0$.
In fact, we can ensure $\|f\| = 1$ and $f(x) = \delta := \operatorname{dist}(x, Y) := \inf_{y \in Y} \|x - y\|$.



(b) If $0 \neq x \in X$ then $\exists f \in X^*$ s.t. $\|f\| = 1$ and $f(x) = \|x\|$.

(c) Bounded linear functionals separate points of X, i.e. for $x \neq y$ in X, there is $f \in X^*$ with $f(x) \neq f(y)$.

(d) For each $x \in X$, define $\hat{x}: X^* \to \mathbb{C}$ by $\hat{x}(f) := f(x)$. Then $\|\hat{x}\| = \|x\|$, so the map $x \mapsto \hat{x} : X \to X^{**}$ is an isometry. We identify X with its image in X^{**} and write $X \subseteq X^{**}$.

Proof. (a) Define $f: Y + \mathbb{C}x \to \mathbb{C}$ by $f(y + \lambda x) = \lambda \cdot \delta$. This is a bdd linear functional with $\|f\| = 1$ because $|f(y + \lambda x)| = |\lambda| \cdot \delta \le |\lambda| \, \|\frac{1}{\lambda} y + x\| = \|y + \lambda x\|$ and $\forall \varepsilon > 0$ we can choose $y \in Y$ so that $\|y + x\| \le \frac{\delta}{1 - \varepsilon}$ (†) hence $z := \frac{1}{\|y + x\|}(y + x)$ has norm 1 and

$$|f(z)| = \frac{1}{\|y+x\|} f(y+x) = \frac{\delta}{\|y+x\|} \overset{(\dagger)}{\geq} \frac{\delta(1-\varepsilon)}{\delta} = 1 - \varepsilon. \qquad \boxed{a}$$